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HAMILTONIAN CYCLES AND PATHS IN VERTEX-TRANSITIVE GRAPHS WITH ABELIAN AND NILPOTENT GROUPS

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In 1968, L. Lovász conjectured that every connected, vertex-transitive graph had a Hamiltonian path. In this paper the following results are proved: (1) If a connected graph has a transitive nilpotent group acting on it, then the graph has a Hamiltonian path; (2) a connected, vertex-transitive graph with a prime power number of vertices has a Hamiltonian path.

1. Introduction

In 1968, L. Lovász made the following conjecture: Every connected, vertex-transitive graph has a Hamiltonian path, [6]. L. Babai proved that every such graph with a prime number of vertices greater than two is Hamiltonian. The Lovász conjecture has been verified for all graphs with $2p$, $3p$, $4p$, $5p$, p^2 and p^3 number of vertices for p a prime. For details see [1]. This paper looks at the Lovasz conjecture from the point-of-view of the automorphism group of the graph, rather than the number of vertices.

All graphs in this paper will have neither loops nor multiple edges. G will always be a connected graph. For any graph H, VH will be the set of vertices of H and EH will be the set of edges of H. Adjacency will be denoted by concatenation, i.e., if $x, y \in \mathrm{VH}$, $xy \in \mathrm{EH}$ means x and y are adjacent in H. Standard names for graphs will be assumed. See [4]. For any set Y, $|Y|$ will denote the cardinality of Y.

Aut H denotes the full automorphism group of H. If $\alpha \in \mathrm{Aut}\, H$ and $x \in \mathrm{VH}$, then $x\alpha$ is the image of x under the action of α. If Γ is a subgroup of Aut H, $x \in \mathrm{VH}$, then Γ_x = stabilizer of x in $\Gamma = \{\alpha \in \Gamma \mid x\alpha = x\}$. The set of images of x under the action of Γ is called the *orbit* of x.

An important, elementary result concerning permutation groups is the following (see [3]):

$$|\Gamma| = |\text{orbit of } x| \cdot |\Gamma_x|.$$

If $\alpha \in \Gamma$, the group generated by $\alpha = \langle\alpha\rangle$ will be denoted by A (α is the only name for an element of Γ that will be so considered). A is always a cyclic group. In general a cyclic group of order n will be denoted Z_n.

A graph H is *vertex transitive* if Aut H has only one orbit. In general if X is any set and Γ is a group of permutation of X, Γ *acts transitively* on X means that Γ has only one orbit. The results of this paper depend upon the nature of certain transitive subgroups Γ of Aut G.

If G is vertex transitive and has a Hamiltonian path, and $x \in \mathrm{VG}$, then G has a Hamiltonian path beginning at x. The main results of this paper are the following:

Theorem 4.1. *Let Γ be transitive on* VG *and nilpotent. Then G has a Hamiltonian path.*

Theorem 5.1. *If G is a vertex transitive graph and* $|\mathrm{VG}| = p^k$, *p a prime, then G has a Hamiltonian path.*

2. Imprimitive group actions

Let X be a finite set and Γ be a transitive group of permutations of X. The action of Γ on X is called *imprimitive* if there exists a partition of X, $X = X_1 \cup X_2 \cup \cdots \cup X_m$, $|X_i| > |$, so that every $\alpha \in \Gamma$ preserves the partition. That is, for all i, $(X_i)\alpha = X_j$ for some j.

The sets X_i are called *blocks of imprimitivity of X for Γ*, and from the above definition it is clear that Γ acts transitively on the set of blocks of imprimitivity of X. A group action is primitive if it is not imprimitive.

Suppose that Γ acts imprimitively on VG. Define a graph C as follows: VC is the set of blocks of imprimitivity of VG: if $c, c' \in \mathrm{VC}$, then $cc' \in \mathrm{EC}$ if and only if there exists vertices $x \in c$, $x' \in c'$ so that $xx' \in \mathrm{EG}$. That is, the vertices of C are the blocks of VG and 2 blocks are adjacent if and only if the sets of vertices are adjacent.

Example 2.1. $G = C_{pq}$, the cycle on p times q vertices. Let $\Gamma = Z_{pq}$, the cyclic group of order pg. Γ acts on VG by 'rotation' in the obvious way. Let Δ be the subgroup of Γ of order p. The q orbits of Δ are blocks of imprimitivity of VG for Γ. The graph C has $\mathrm{VC} = q$ and is isomorphic to C_q, the cycle on q vertices.

Imprimitive actions arise in the following way (this result appears as Theorem 5.6.1 of [3]):

Lemma 2.1. *Let Γ be a transitive group of permutations of a finite set X. Let Δ be a subgroup of Γ so that for some $x \in X$, $\Gamma_x \subsetneq \Delta \subsetneq \Gamma$. Then the orbit of Δ containing x is a block of imprimitivity of X for Γ. The translates of this orbit by Γ form a set of* *blocks. Further, every set of blocks of imprimitivity is the set of translates of such an orbit of such a subgroup of Γ.*

Lemma 2.2. *Let Γ be a transitive group of permutations of a finite set X. Let Δ be a normal subgroup of Γ. If Δ is not transitive on X, then the orbits of Δ form a set of blocks of imprimitivity for Γ.*

Lemma 2.3. *Let Γ be a transitive subgroup of* Aut G. *Let Δ be a normal subgroup of Γ. Let C be the graph as defined above on the orbits of Δ. Then Γ/Δ is a transitive subgroup of* Aut C.

The preceding results can be used to construct an induction tool. That tool is Theorem 2.1 below.

Theorem 2.1. *Let Γ be an imprimitive, transitive subgroup of* Aut G, *with graph C as defined above. If each $c \in C$ has a Hamiltonian path, considered as a subgraph of G, and C itself has a Hamiltonian path, then G has a Hamiltonian path.*

Proof. Let $c^1 c^2 \ldots c^m$ be a Hamiltonian path in C. c^i is the translate of an orbit of a subgroup of Γ, so for every vertex in c^i there exists a Hamiltonian path beginning at that vertex. Further, if $c^i c^{i+1} \in \mathrm{E}C$, then every $x^i_j \in c^i$ is adjacent with some $x^{i+1}_k \in c^{i+1}$. This follows since there is an element of Δ mapping $z^i \in c^i$ to x^i_j for every x^i_j. Then $x^1_1 - - - x^1_n x^2_{i_2} - - - x^2_{j_2} x^3_{i_3} - - - \ldots x^m_{i_m} - - - x^m_{j_m}$ is a Hamiltonian path in G, where - - - denotes the appropriate Hamiltonian path in each c^i. □

Example 2.2. Let P be the Petersen graph as labeled in Fig. 1. Let Γ be the subgroup of Aut P generated by the automorphisms $\alpha = (1\ 2\ 3\ 4\ 5)\ (a\ c\ e\ b\ d)$

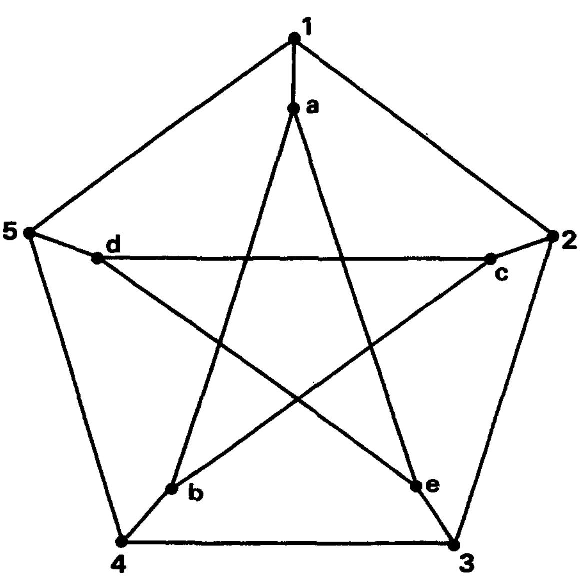

Fig. 1. P: The Petersen graph; $\alpha = (1\ 2\ 3\ 4\ 5)(a\ c\ e\ b\ d)$; $\beta = (1a)(2b\ 5e)(3c\ 4d)$; $\beta^2 = (2\ 5)(3\ 4)(c\ d)(b\ e)$.

and $\beta=(1a)(2b\ \ 5e)(3c\ \ 4d)$. α maps the outer (numeric) and inner (alphabetic) cycles to themselves, and β exchanges them. Hence these cycles are blocks of imprimitivity of VP for Γ. Each cycle is Hamiltonian. The graph C is isomorphic with K_2. Hence P has a Hamiltonian path, but as is well known P is not Hamiltonian. The diehedral group, D_5, generated by α and β^2 is the subgroup of Γ specified in Lemma 2.1.

3. Transitive abelian groups

The *center* of Γ, denoted $Z(\Gamma)$, is the subgroup $\{\alpha\in\Gamma \mid \text{for all } \beta\in\Gamma,\ \alpha\beta=\beta\alpha\}$. That is, the center of Γ consists of all elements of Γ which commute with every element of Γ. $Z(\Gamma)$ is a normal subgroup of Γ. If $\alpha\in Z(\Gamma)$, then $A=\langle\alpha\rangle=$ the group generated by α is also a normal subgroup of Γ. Thus, given $\alpha\in Z(\Gamma)$, $\alpha\neq 1$, the graph C can be defined as in Section 2.

Theorem 3.1. *Let $\alpha\in Z(\Gamma)$, $\alpha\neq 1$. If C has a Hamiltonian path and there exists $x\in \mathrm{VG}$ so that x is adjacent to $x\alpha$ in G, then G is Hamiltonian.*

Proof. The vertices of C are the orbits of α. Let $\mathrm{VC}=\{c^1,c^2,\ldots,c^m\}$, and suppose that $c^i=(x_1^i,x_2^i,\ldots,x_r^i)$ is an orbit of α. Without loss of generality suppose $c^1c^2\ldots c^m$ is a Hamiltonian path in C. Since orbits may be written beginning with any element, and by the definition of the graph C, we may assume that $x_1^ix_1^{i+1}\in\mathrm{EG}$ for $i=1,\ldots,m-1$. Then $x_i^k\alpha^k$ is adjacent to $x_1^{i+1}\alpha^k$ for all k, that is $x_k^ix_k^{i+1}$ for all $k=1,\ldots,r$. Suppose that $x_1^ix_2^i\in\mathrm{EG}$, that is x_1^i is adjacent to $x_1^i\alpha$. Then $x_k^ix_{k+1}^i\in\mathrm{EG}$ for all $k=1,\ldots,r$, where $r+1$ is taken modulo r. Since Γ is transitive on VG, there exists $\beta_j\in\Gamma$ so that $x_1^i\beta_j=x_1^j$. $\alpha\in Z(\Gamma)$ implies that $\alpha\beta=\beta\alpha$. Thus $x_1^i\beta\alpha=x_1^j\alpha=x_2^j=x_1^i\alpha\beta=x_2^i\beta$. Thus $x_1^jx_2^j\in\mathrm{EG}$ also. Thus, for all $i=1,\ldots,m$ and for all $k=1,\ldots,r$, $x_k^ix_{k+1}^i\in\mathrm{EG}$, and for $i\neq m$, $x_k^ix_k^{i+1}\in\mathrm{EG}$. See Fig. 2.

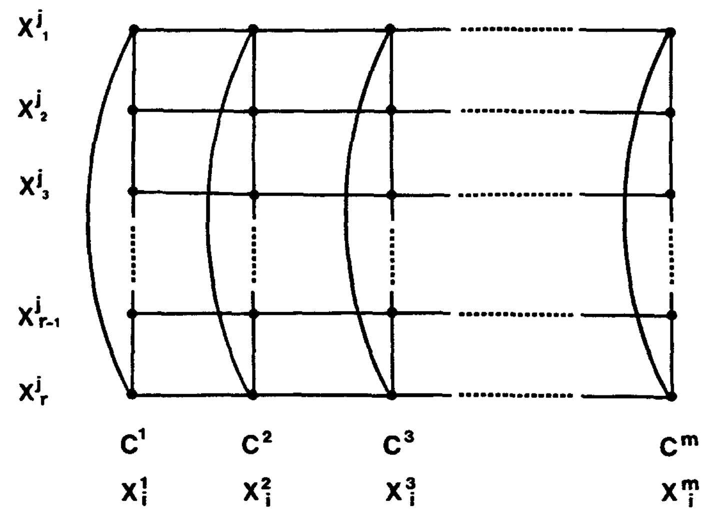

Fig. 2.

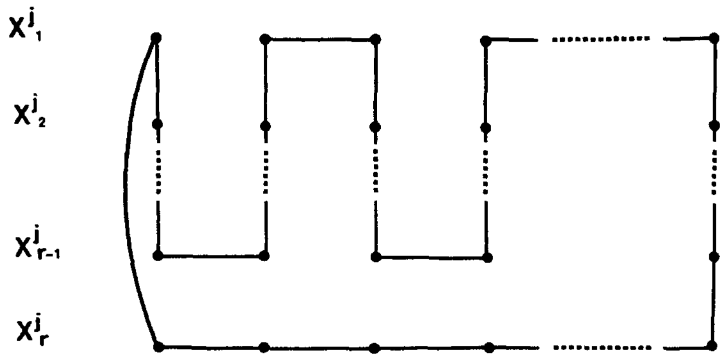

Fig. 3. m is odd.

Case 1: m is odd. $x_1^1x_2^1\ldots x_{r-1}^1x_{r-1}^2x_{r-2}^2\ldots x_1^2x_1^3x_2^3\ldots x_1^{m-1}x_1^mx_2^m\ldots x_{r-1}^mx_r^mx_r^{m-1}\ldots x_r^2x_n^1x_1^1$ is a Hamiltonian cycle. See Fig. 3.

Case 2: m is even. $x_1^1x_2^1\ldots x_r^1x_r^2x_{r-1}^2\ldots x_2^2x_2^3x_3^3\ldots x_r^3\ldots x_r^{m-1}x_r^mx_{r-1}^m\ldots x_1^mx_1^{m-1}\ldots x_1^1$ is a Hamiltonian cycle. See Fig. 4. □

The following known results follow easily from Theorem 3.1:

Corollary 3.1. *Let Γ be transitive on* VG *and abelian. Then $G=K_1$, K_2, or G is Hamiltonian.*

Corollary 3.2. *If Γ is transitive and abelian on* VG, *then every edge in* EG *lies on a Hamiltonian cycle.*

The above results include as a special case the results of Chen and Quimpo [2], Lee [5], and Marusic [7]:

Corollary 3.3. *If G is a connected Cayley graph of an abelian group, then G is Hamiltonian and every edge G lies in a Hamiltonian cycle.*

Example 3.1. The following graphs all have non-abelian automorphism groups which have transitive abelian subgroups: K_n, $n\geqslant 3$; C_n^r, $n\geqslant 3$; $C_a\times C_b$, a and $b\geqslant 3$; $K_{m,m,\ldots,m}$.

Example 3.2. There are vertex transitive Hamiltonian graphs that do not have

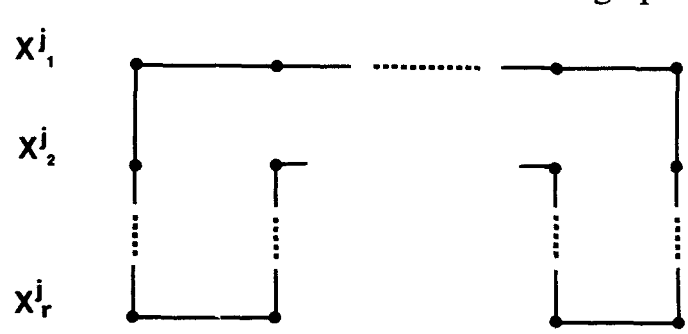

Fig. 4. m is even.

abelian transitive groups acting on them. Let P be the Petersen graph, and consider $G = P \times K_2$.

G is Hamiltonian and P is not. By considering the 4 cycles of G it is easy to see that $\mathrm{Aut}\, G = \mathrm{Aut}\, P + Z_2$, where $+$ means 'direct sum'. If Δ is transitive on VG, then Δ / Z_2 is transitive on P. By Corollary 3.1, Δ / Z_2 is not abelian, so Δ is also not abelian.

4. Transitive nilpotent groups

A group Γ is called *nilpotent* if there exists a set of normal subgroups of Γ, $\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_n = 1$ so that $\Gamma_{i+1} \subseteq \Gamma_i$ and $\Gamma_i / \Gamma_{i+1} \subseteq Z(\Gamma / \Gamma_{i+1})$ for all $i = 1, \ldots, h-1$.

Abelian groups are trivially nilpotent. For the purposes of this paper only 3 basic properties of nilpotent groups are needed. See [3]:

(1) Subgroups and quotient groups of a nilpotent group are nilpotent;
(2) Every maximal subgroup of a nilpotent group is normnal;
(3) Groups of prime power order are nilpotent.

Theorem 4.1. *Suppose that* Aut G *has a transitive nilpotent subgroup. Then* G *has a Hamiltonian path.*

Proof. Choose Γ to be a minimal transitive nilpotent subgroup of Aut G. The proof is by induction on $|\mathrm{VG}|$. If $|\mathrm{VG}|$ is prime the result is clear. If $|\mathrm{VG}|$ is not prime, choose x, $y \in \mathrm{VG}$ with $xy \in \mathrm{EG}$ and $\alpha \in \Gamma$ so that $x\alpha = y$. If $A = \langle \alpha \rangle$ is transitive on VG, then by Corollary 3.1 G is Hamiltonian. Otherwise A is contained in a maximal subgroup M of Γ. M is a normal subgroup of Γ, and not transitive. The orbits of M contain edges of G since $xy \in \mathrm{EG}$ and x and y are in an orbit of A. Therefore the components of the orbits of M form a set of connected blocks of imprimitivity of VG for Γ. Each component is a connected vertex transitive graph, and is acted on transitively by a subgroup of M.

Define C as usual on this set of blocks of imprimitivity. Γ / M acts on VC. Since subgroups and quotient groups of nilpotent groups are nilpotent, by induction each block of imprimitivity has a Hamiltonian path and the graph C has a Hamiltonian path. By Theorem 2.1, G has a Hamiltonian path. □

5. Graphs on a prime power number of vertices

Let $|\Gamma| = p^l b$ with p a prime and p not dividing b. A subgroup Δ of Γ with $|\Delta| = p^l$ is called a p-Sylow subgroup of Γ. p-Sylow subgroups exist for all primes p and all groups Γ. All p-Sylow subgroups are conjugate, that is, isomorphic via an inner automorphism of Γ. Further, every subgroup of Γ of order p^k is contained in some p-Sylow subgroup of Γ. See [3] for details.

Lemma 5.1. *If* $|VG| = p^k$, *then every* p-*Sylow subgroup of* Aut G *acts transitively on* VG.

Theorem 5.1. *If* G *is a connected, vertex transitive graph and* $|VG| = p^k$, p *a prime, then* G *has a Hamiltonian path.*

Proof. Let Γ be a p-Sylow subgroup of Aut G. Γ is transitive by Lemma 5.1 and nilpotent. By Theorem 4.1, G has a Hamiltonian path. □

It is unknown if this result can be strengthened. Every known connected vertex transitive graph with a prime power number of vertices greater than or equal to 3 is Hamiltonian.

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